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SYMMETRIC FLOW OVER THIN GENERALIZED AXISYMMETRIC CONICAL BODIES IN A SUPERSONIC INVISCID GAS STREAM

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ABSTRACT. Generalized axisymmetric flow over cones is analyzed. The analysis is limited to the case where the body is within the Mach cone. An asymptotic equation for the meridianal cross section is presented for a sharp tipped cone. The solution of this equation is obtained numerically, and the resulting plots of β versus α are shown graphically.

The linearized theory of supersonic conical flows of an inviscid gas can be applied to problems other than those of gas dynamics. D. D. Ivlev [1] showed that the resulting linearized theory of supersonic gas flows can be transferred directly to the theory of penetration of sharp tipped bodies into an ideal plastic medium. /39[1]

The theory of linearized conical flows of A. Buzeman [2] was generalized by M. D. Khaskind and S. V. Falkovich [3]. As we know, conical flows are characterized by the fact that the velocity along the rays exiting from a certain flow pole are constant. In generalized conical flows, these velocities are proportional to whole, positive powers of the distance from the pole, i.e. from the tip of a sharp tipped body around which the flow is occurring. Generalized conical flows also include flows produced by superimposition of velocity fields of these flows. In this work, we will analyze a particular case of axisymmetrical generalized conical flows. For this purpose, we will use the general solution of the problem of generalized conical flows in the form presented in [4, 5].

We will present the results from [4] required for the following without proof.

Initial Equations

Suppose the main, unperturbed flow approaching the generalized conical body with tip at zero has velocity W at infinity, directed along the z axis (Figure 1). In the following, we will analyze only the case when the body in the flow is located completely within the Mach cone with its peak corresponding to the tip of the body. The velocity projections in the gas flow disturbed by the body will be represented as u, v and w. These projections are considered small in comparison with W. The projections of the total flow velocities will

[1] Numbers in the margin indicate pagination in the foreign text.

be u, v, W + w.

Figure 1 shows the plane z = 1, passing through point 0. This plane contains the $O\xi$ and $O\eta$ axes, parallel to the O_1x, O_1y axes. We can look upon ξ and η as dimensionless coordinates $\xi = x/z$, $\eta = y/z$. The cross sections of the body and of the Mach cone in plane ξ, η are circles. /40

Let us introduce the new, independent variables using the formulas:

$$x = -r\frac{\cos\sigma}{\operatorname{sh}\delta}; \quad y = -r\frac{\sin\sigma}{\operatorname{sh}\delta}; \quad Az = -r\operatorname{cth}\delta. \tag{1}$$

Here $A = (M^2 - 1)^{-1/2}$, M is the Mach number, σ is the angle of the radius of the vector in plane ξ, η with the ξ axis. By excluding σ and δ from the equations of (1), we can find

$$r^2 = A^2 z^2 - x^2 - y^2. \tag{2}$$

The potential of the velocities of the perturbed flow Φ should satisfy the equation

$$\left(\frac{\partial^2 \Phi}{\partial \sigma^2} + \frac{\partial^2 \Phi}{\partial \delta^2}\right)$$
$$\frac{1}{\operatorname{sh}^2\delta} \cdot \frac{\partial}{\partial r}\left[r^2 \frac{\partial \Phi}{\partial r}\right].$$

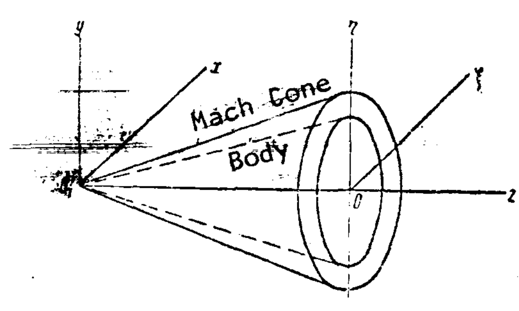

Figure 1

The solution of this equation has the form

$$\Phi = \Phi_1 + \Phi_2 + \ldots + \Phi_n + \ldots, \tag{3}$$

where $\Phi_n = r^n \varphi_n(\delta, \sigma)$ and the φ_n functions are satisfied by the equation

$$\frac{\partial^2 \varphi_n}{\partial \sigma^2} + \frac{\partial^2 \varphi_n}{\partial \delta^2} - \frac{n(1+n)}{\operatorname{sh}^2\delta}\varphi_n = 0. \tag{4}$$

In [4], the following solution is presented for (4), dependent on the arbitrary harmonic function $F(\sigma, \delta)$:

$$\varphi_n = \mathrm{sh}^{n+1}\,\delta\left(\frac{1}{\mathrm{sh}\,\delta}\cdot\frac{\partial}{\partial\delta}\right)^n \frac{F(\sigma,\delta)}{\mathrm{sh}\,\delta}. \tag{5}$$

Axisymmetrical Flow Case

In the case of axisymmetrical flow, function ϕ_n does not depend on σ. Equation (4) in this case takes on the form

$$\frac{d^2\varphi_n}{d\delta^2} - \frac{n(1+n)}{\mathrm{sh}^2\,\delta}\varphi_n = 0. \tag{6}$$

The harmonic function, which is independent of σ, has the form $F(\delta) = C\delta + C_1$. Let us prove that it is correct to assume $C_1 = 0$. For this purpose, let us study the behavior of the solution as we approach the Mach cone, determined by the equation

$$r^2 = A^2 z^2 - x^2 - y^2 = 0. \tag{7}$$

It follows from (1) that as we approach the Mach cone, $\delta \to 0$, the order of /41
the approach of δ and r to 0 being identical. Substituting $F = C\delta + C_1$ in (5), it is not difficult to see that as $\delta \to 0$

$$\varphi_n = \mathrm{sh}^{n+1}\,\delta\left(\frac{1}{\mathrm{sh}\,\delta}\cdot\frac{\partial}{\partial\delta}\right)^n \frac{C\delta + C_1}{\mathrm{sh}\,\delta} = CO(\delta^{n+1}) + C_1 O(\delta^{-n}).$$

From this it follows that

$$\Phi_n = r^n\varphi_n = CO(\delta^{2n+1}) + C_1 O(1).$$

But on the surface of the Mach cone, in the case when the body is located entirely within the Mach cone, the velocity potential of the perturbed flow must be equal to zero. Therefore, it is necessary that $C_1 = 0$ and, finally,

$$\varphi_n = C\,\mathrm{sh}^{n+1}\,\delta\left(\frac{1}{\mathrm{sh}\,\delta}\cdot\frac{\partial}{\partial\delta}\right)^n \frac{\delta}{\mathrm{sh}\,\delta}. \tag{8}$$

Form of Tip of Body

Suppose $R = \sqrt{x^2 + y^2}$ and $t = R/Az$. In the area of the tip 0, the value of t is low. It follows from (1) that

$$\delta = \frac{1}{2} \ln \frac{Az - r}{Az + r}; \quad \operatorname{sh} \delta = -\frac{r}{R}; \quad \operatorname{cth} \delta = -\frac{Az}{r}. \tag{9}$$

Further, with low t, we can produce the following asymptotic formulas from (2) and (9):

$$r \approx Az; \quad \delta \approx \ln \frac{R}{2Az} = \ln (t/2); \quad d\delta \approx dt/t.$$

Now equation (8) can be written in the form

$$\varphi_n = Ct^{-n-1} \left(t^2 \frac{d}{dt} \right)^n t \ln \frac{t}{2}.$$

From this it is easy to see that where $t \to 0$

$$\varphi_n = O(\ln t) \approx \ln t \cdot \text{const}. \tag{10}$$

Furthermore,

$$\Phi_n = r^n \varphi_n = O\left[(Az)^n \ln \frac{R}{Az} \right],$$

from which[1] /42

$$\frac{\partial \Phi_n}{\partial R} = \frac{z^n}{R} \cdot \text{const}. \tag{11}$$

Considering that $W + \partial \Phi_n / \partial z \approx W$, the differential equation for the flow lines has the form

[1] In those instances when only the order of magnitude is important, the same symbol const will be used for various constants.

$$\frac{dR}{dz} = \frac{\partial \Phi_n}{W \partial R}.$$

Substituting $\partial\Phi_n/\partial R$ from (11) into this equation, we can find

$$\frac{dR}{dz} = \frac{z^n}{R} \cdot \text{const.} \tag{12}$$

Since the tip of the body is at point Z = 0,R = z, it is easy to produce the asymptotic equation for the meridianal cross section of the body from (12):

$$R = z^{\frac{n+1}{2}} \cdot \text{const.} \tag{13}$$

Case n = 1 corresponds to the well known flow about a cone [2]. Below, as an example, we will analyze in detail the case n = 2.

Particular Case n = 2

In the particular case n = 2, by differentiating in (8), we can easily produce

$$\varphi_2 = C[-3 \operatorname{cth} \delta - \delta + 3\delta \operatorname{cth}^2 \delta].$$

Substituting here the value of δ and coth δ from (9), and multiplying ϕ_2 by r^2, we can find

$$\Phi_2 = C\left[3Azr + \frac{2A^2z^2 + R^2}{2} \ln \frac{Az - r}{Az + r}\right]. \tag{14}$$

From which

$$\frac{\partial \Phi_2}{\partial R} = C\left[\frac{2Azr}{R} + R \ln \frac{Az - r}{Az + r}\right] \tag{15}$$

and

$$w = \frac{\partial \Phi_2}{\partial z} = C\left[4Ar + 2zA^2 \ln\frac{Az - r}{Az + r}\right]. \tag{16}$$

The differential equation for the flow lines will have the form /43

$$\frac{dR}{dz} = \frac{C}{W}\left(\frac{2Azr}{R} + R\ln\frac{Az - r}{Az + r}\right). \tag{17}$$

The body around which the flow occurs will correspond to the solution of the equation satisfying the initial condition

$$R\,|_{z=0} = 0. \tag{18}$$

If we introduce the new variables:

$$\alpha = \frac{C}{W} z; \quad \beta = \frac{C}{AW} R, \tag{19}$$

then equation (17) is reduced to an equation not containing the parameters in explicit form:

$$\frac{d\beta}{d\alpha} = \frac{2\alpha\sqrt{\alpha^2 - \beta^2}}{\beta} - 2\beta\ln\frac{\alpha + \sqrt{\alpha^2 - \beta^2}}{\beta}. \tag{20}$$

If $\beta = f(\alpha)$ is the solution of equation (20), satisfying the condition $\beta\,|_{\alpha = 0} = 0$, the equation for the meridianal section of the body has the form

$$R = \frac{AW}{C} f\left(\frac{cz}{W}\right). \tag{21}$$

It is easy to see that the asymptotic solution of equation (20) satisfying the initial condition in the area of the tip of the body has the form

$$\beta = \frac{2}{\sqrt{3}}\alpha^{3/2}, \tag{22}$$

which is in agreement with (13). Equation (20) was integrated by the Runge-Kutta numerical method. The initial values of quantity β were determined from asymptotic formula (22). The results of the calculation are presented in the table below. Figure 2 shows the curve constructed from the results of calculation for low values of α. In order to investigate the behavior of the integral curve over the broadest possible interval during numerical integration, we selected a large step (X = 0.1). The results are shown on Figure 3. The equation for the Mach cone (7) in coordinates α and β becomes the following: $\alpha^2 = \beta^2$ (straight dotted line on Figures 2 and 3). As $\alpha \to \infty$, i.e. as $z \to \infty$, the meridianal cross section of the body is made parallel to the generatrix of the Mach cone in the same meridianal plane. However, the angle between the generatrix of the Mach cone and the z axis is arbitrary, whereas the angle of the body surface with the z axis, in correspondence with the main requirements of the linearized theory, must be small. Therefore, the numerical results produced have physical meaning only for the bow portion of the body as long as it can be considered thin. Since the flow is supersonic, the stern portion of the body can be cut off without changing the flow about the bow portion. Pressure p is determined by the linearized Bernoulli integral [see (16)] /45

$$p = -\rho\varphi W = -\rho W C\left[4Az + 2zA^2 \ln\frac{Az - r}{Az + r}\right], \tag{23}$$

where ρ is the gas density. From this, the drag of the bow portion of the body can be calculated using the formula

$$X = \frac{8\pi\rho A^4 W^2}{C^2} F(\alpha), \tag{24}$$

where

$$F(\alpha) = -\int_0^{\alpha} \beta\left[\sqrt{\alpha^2 - \beta^2} - \alpha \ln\frac{\alpha + \sqrt{\alpha^2 - \beta^2}}{\beta}\right]\frac{d\beta}{d\alpha}\, d\alpha.$$

Figure 4 shows the dependence $F(\alpha)$, which is constructed from the data shown in the table.

α	β	$F(\alpha)$	β asympt
0,001	$0,364314\cdot 10^{-4}$	$0,157162\cdot 10^{-11}$	$0,365148\cdot 10^{-4}$
0,002	$0,102839\cdot 10^{-3}$	$0,221348\cdot 10^{-10}$	$0,103280\cdot 10^{-3}$
0,003	$0,188583\cdot 10^{-3}$	$0,103446\cdot 10^{-9}$	$0,189737\cdot 10^{-3}$
0,004	$0,289837\cdot 10^{-3}$	$0,307703\cdot 10^{-9}$	$0,292119\cdot 10^{-3}$
0,005	$0,404381\cdot 10^{-3}$	$0,714761\cdot 10^{-9}$	$0,408248\cdot 10^{-3}$
0,006	$0,530710\cdot 10^{-3}$	$0,142023\cdot 10^{-8}$	$0,536656\cdot 10^{-3}$
0,007	$0,667714\cdot 10^{-3}$	$0,253403\cdot 10^{-8}$	$0,676264\cdot 10^{-3}$
0,008	$0,814533\cdot 10^{-3}$	$0,417916\cdot 10^{-8}$	$0,826236\cdot 10^{-3}$
0,009	$0,970468\cdot 10^{-3}$	$0,649078\cdot 10^{-8}$	$0,985900\cdot 10^{-3}$
0,010	$0,113194\cdot 10^{-2}$	$0,961529\cdot 10^{-8}$	$0,115470\cdot 10^{-2}$
0,011	$0,130747\cdot 10^{-2}$	$0,137097\cdot 10^{-7}$	$0,133216\cdot 10^{-2}$
0,012	$0,148762\cdot 10^{-2}$	$0,189407\cdot 10^{-7}$	$0,151789\cdot 10^{-2}$
0,013	$0,167504\cdot 10^{-2}$	$0,254845\cdot 10^{-7}$	$0,171153\cdot 10^{-2}$
0,014	$0,186938\cdot 10^{-2}$	$0,335259\cdot 10^{-7}$	$0,191276\cdot 10^{-2}$
0,015	$0,207037\cdot 10^{-2}$	$0,432581\cdot 10^{-7}$	$0,212132\cdot 10^{-2}$
0,016	$0,227774\cdot 10^{-2}$	$0,548820\cdot 10^{-7}$	$0,233695\cdot 10^{-2}$
0,017	$0,249126\cdot 10^{-2}$	$0,686060\cdot 10^{-7}$	$0,255942\cdot 10^{-2}$
0,018	$0,271070\cdot 10^{-2}$	$0,846455\cdot 10^{-7}$	$0,278854\cdot 10^{-2}$
0,019	$0,293586\cdot 10^{-2}$	$0,103223\cdot 10^{-6}$	$0,302412\cdot 10^{-2}$
0,020	$0,316658\cdot 10^{-2}$	$0,124566\cdot 10^{-6}$	$0,326598\cdot 10^{-2}$
0,021	$0,340267\cdot 10^{-2}$	$0,148909\cdot 10^{-6}$	$0,351397\cdot 10^{-2}$
0,022	$0,364398\cdot 10^{-2}$	$0,176494\cdot 10^{-6}$	$0,376793\cdot 10^{-2}$
0,023	$0,389036\cdot 10^{-2}$	$0,207565\cdot 10^{-6}$	$0,402773\cdot 10^{-2}$
0,024	$0,414168\cdot 10^{-2}$	$0,242375\cdot 10^{-6}$	$0,429324\cdot 10^{-2}$
0,025	$0,439780\cdot 10^{-2}$	$0,281178\cdot 10^{-6}$	$0,456435\cdot 10^{-2}$
0,026	$0,465861\cdot 10^{-2}$	$0,324236\cdot 10^{-6}$	$0,484093\cdot 10^{-2}$
0,027	$0,492400\cdot 10^{-2}$	$0,371814\cdot 10^{-6}$	$0,512288\cdot 10^{-2}$
0,028	$0,519385\cdot 10^{-2}$	$0,424183\cdot 10^{-6}$	$0,541011\cdot 10^{-2}$
0,029	$0,546806\cdot 10^{-2}$	$0,481616\cdot 10^{-6}$	$0,570251\cdot 10^{-2}$
0,030	$0,574653\cdot 10^{-2}$	$0,544392\cdot 10^{-6}$	$0,599999\cdot 10^{-2}$
0,031	$0,602918\cdot 10^{-2}$	$0,612792\cdot 10^{-6}$	$0,630248\cdot 10^{-2}$
0,032	$0,631592\cdot 10^{-2}$	$0,687102\cdot 10^{-6}$	$0,660988\cdot 10^{-2}$
0,033	$0,660666\cdot 10^{-2}$	$0,767612\cdot 10^{-6}$	$0,692213\cdot 10^{-2}$
0,034	$0,690133\cdot 10^{-2}$	$0,854613\cdot 10^{-6}$	$0,723914\cdot 10^{-2}$
0,035	$0,719984\cdot 10^{-2}$	$0,948402\cdot 10^{-6}$	$0,756085\cdot 10^{-2}$
0,036	$0,750212\cdot 10^{-2}$	$0,104928\cdot 10^{-5}$	$0,788719\cdot 10^{-2}$
0,037	$0,780811\cdot 10^{-2}$	$0,115754\cdot 10^{-5}$	$0,821810\cdot 10^{-2}$
0,038	$0,811774\cdot 10^{-2}$	$0,127350\cdot 10^{-5}$	$0,855350\cdot 10^{-2}$
0,039	$0,843094\cdot 10^{-2}$	$0,139746\cdot 10^{-5}$	$0,889335\cdot 10^{-2}$
0,040	$0,874766\cdot 10^{-2}$	$0,152973\cdot 10^{-5}$	$0,923759\cdot 10^{-2}$
0,041	$0,906782\cdot 10^{-2}$	$0,167062\cdot 10^{-5}$	$0,958616\cdot 10^{-2}$
0,042	$0,939138\cdot 10^{-2}$	$0,182015\cdot 10^{-5}$	$0,993900\cdot 10^{-2}$
0,043	$0,971828\cdot 10^{-2}$	$0,197954\cdot 10^{-5}$	$0,102981\cdot 10^{-1}$
0,044	$0,100485\cdot 10^{-1}$	$0,214820\cdot 10^{-5}$	$0,106573\cdot 10^{-1}$
0,045	$0,103819\cdot 10^{-1}$	$0,232675\cdot 10^{-5}$	$0,110227\cdot 10^{-1}$
0,046	$0,107185\cdot 10^{-1}$	$0,251553\cdot 10^{-5}$	$0,113921\cdot 10^{-1}$
0,047	$0,110582\cdot 10^{-1}$	$0,271486\cdot 10^{-5}$	$0,117656\cdot 10^{-1}$
0,048	$0,114010\cdot 10^{-1}$	$0,292505\cdot 10^{-5}$	$0,121431\cdot 10^{-1}$
0,049	$0,117469\cdot 10^{-1}$	$0,314645\cdot 10^{-5}$	$0,125246\cdot 10^{-1}$
0,050	$0,120958\cdot 10^{-1}$	$0,337938\cdot 10^{-5}$	$0,129099\cdot 10^{-1}$

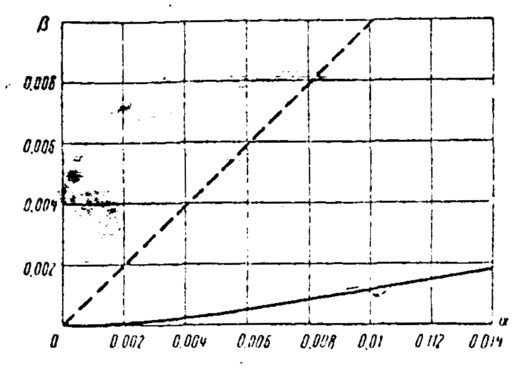

Figure 2

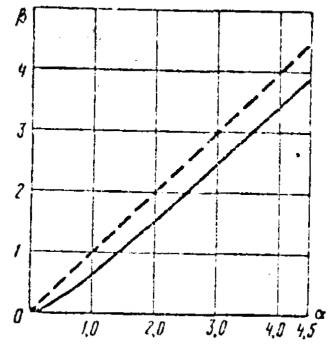

Figure 3

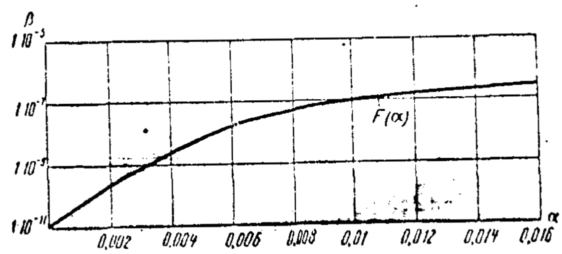

Figure 4

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